C3 Specimen (MA)

Q1(a) $f(x) = 1$: $|x-2| - 3 = 1$

$|x-2| = 4$

two solutions : $x - 2 = 4$ or $-(x-2) = 4$

$\boxed{x = 6}$ $\quad$ $2 - x = 4$

$\boxed{x = -2}$

b) $g(x) = (x-2)^2 - 4 + 11$

$= (x-2)^2 + 7$ $\therefore$ turning point: $(2, 7)$

$\therefore$ range is $\boxed{7 \leq g(x)}$

c) $f(-1) = |-1-2| - 3 = 0$

$gf(-1) = g(0) = \boxed{11}$

Q2a) $f(2) = 2^3 - 2(2) - 5 = -1$

$f(3) = 3^3 - 2(3) - 5 = 16$

Change of sign between $x = 2$ and $x = 3$ $\therefore$ a root α lies in the interval $[2, 3]$

b) $x_1 = \sqrt{2 + \frac{5}{2}} = 2.121$

similarly, $x_2 = 2.087$

$x_3 = 2.097$

$x_4 = 2.094$

c) $f(2.09455) = -0.00001\ldots$ ⎫ change in sign hence $\alpha = 2.0946$ to 5 s.f.

$f(2.09465) = 0.001099$ ⎭

Q3a) $\cos\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = \cos\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\sin\frac{\theta}{2}$

$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= 1-\sin^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= 1 - 2\sin^2\frac{\theta}{2}$

$\square$

b) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

$\therefore \sin\theta - \cos\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + 2\sin^2\frac{\theta}{2} - 1$

$\therefore 1 + \sin\theta - \cos\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + 2\sin^2\frac{\theta}{2}$

$= 2\sin\frac{\theta}{2}\left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right]$

$\square$

c) $2\sin\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right] = 0$

$\sin\frac{\theta}{2} = 0$

$\frac{\theta}{2} = 0$

$\boxed{\theta = 0}$

$\cos\frac{\theta}{2} + \sin\frac{\theta}{2} = 0$

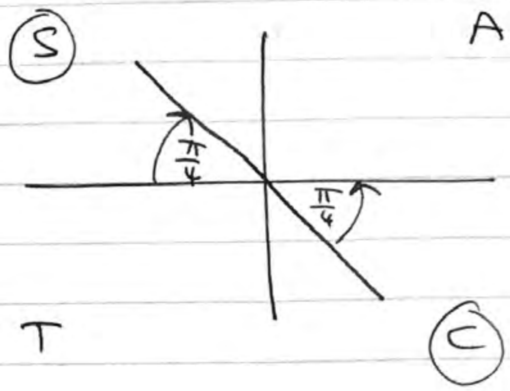
$\div \cos\frac{\theta}{2}$: $1 + \tan\frac{\theta}{2} = 0$

$\tan\frac{\theta}{2} = -1$

$\frac{\theta}{2} = \tan^{-1}(-1)$

$$\frac{\theta}{2} = \frac{-\pi}{4}$$

$$0 \leq \frac{\theta}{2} < \pi$$



$$\frac{\theta}{2} = \frac{3\pi}{4}$$

$$\boxed{\theta = \frac{3\pi}{2}}$$

Q4a) $f(x) = \dfrac{x(x-1)}{(x-1)} + \dfrac{3}{(x-1)} + \dfrac{-12}{(x+3)(x-1)}$

$$= \frac{x(x-1)(x+3) + 3(x+3) - 12}{(x-1)(x+3)}$$

$$= \frac{x(x^2+2x-3) + 3x + 9 - 12}{(x-1)(x+3)}$$

$$= \frac{x^3 + 2x^2 - 3x + 3x - 3}{(x-1)(x+3)}$$

$$= \frac{x^3 + 2x^2 - 3}{(x-1)(x+3)} = \frac{(x^2+3x+3)\cancel{(x-1)}}{\cancel{(x-1)}(x+3)}$$

$$= \boxed{\frac{x^2+3x+3}{x+3}}$$

b) $f(x) = \dfrac{x^2+3x+3}{x+3}$ QUOTIENT RULE

$u = x^2+3x+3 \longrightarrow u' = 2x+3$

$v = x+3 \longrightarrow v' = 1$

$$f'(x) = \frac{vu'-uv'}{v^2} = \frac{(x+3)(2x+3)-x^2-3x-3}{(x+3)^2}$$

$$= \frac{2x^2+9x+9-x^2-3x-3}{(x+3)^2} = \frac{22}{25}$$

$$= \frac{x^2+6x+6}{(x+3)^2} = \frac{22}{25}$$

$$\therefore \quad x^2+6x+6 = \frac{22}{25}(x^2+6x+9)$$

$$\Rightarrow \quad x^2+6x+6 = \frac{22}{25}x^2 + \frac{132x}{25} + \frac{198}{25}$$

$\times 25$

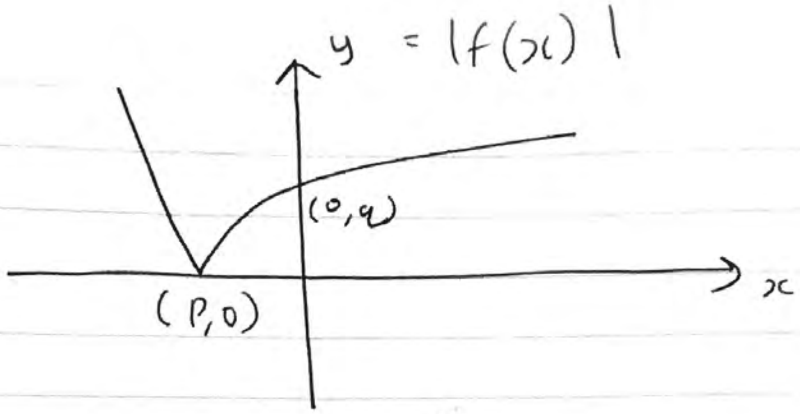
$$\Rightarrow \quad 25x^2+150x+150 = 22x^2+132x+198$$

$$\Rightarrow \quad 3x^2+18x-48 = 0$$

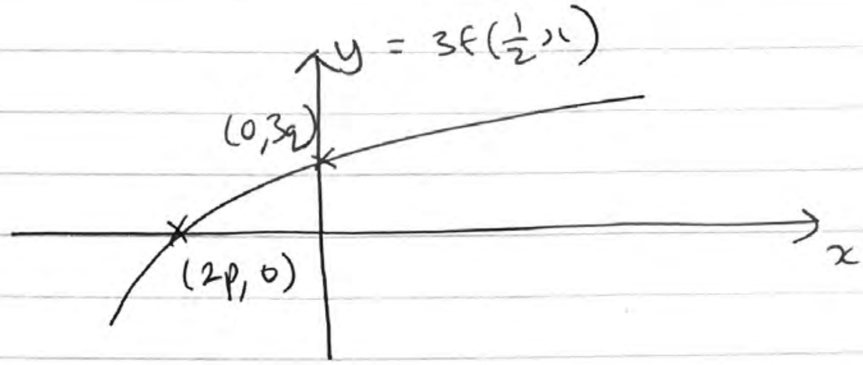
$$\Rightarrow \quad x^2+6x-16 = 0 \longrightarrow (x+8)(x-2) = 0$$

$\therefore \boxed{x = 2}$ as $x > 1$.

Q5a) i)



ii)



b) $\underline{x=0}$: $\boxed{q = 3\ln 3}$

c) $3\ln(2x+3) = 0$

$e^0 = 2x + 3 = 1$

$x = \boxed{-1 = p}$

d) $\frac{dy}{dx} = 3\left[\frac{2}{2x+3}\right] = \frac{6}{2x+3}$ //

at $x = -1$, $\frac{dy}{dx} = \frac{6}{1} = 6$ //

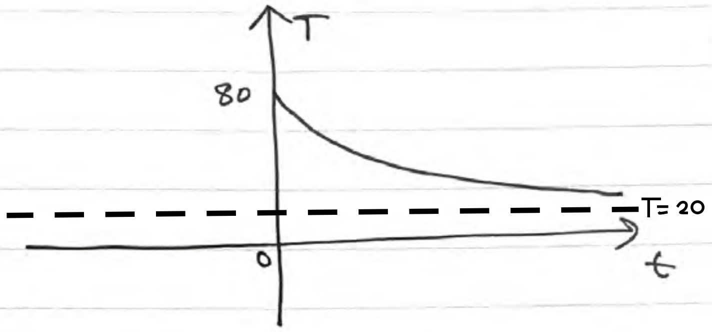
$\Rightarrow \quad y - 0 = 6(x+1) \longrightarrow \boxed{y = 6x + 6}$

Q6a) $T = 20 + 60e^{-0.1t}$

$\underline{t=0}$: $T = 20 + 60 = \boxed{80}$

b) Because $[60e^{-0.1t}] > 0$.

c)



$(t \geq 0)$.

d) $60 = 20 + 60e^{-0.1t}$

$\frac{4}{6} = e^{-0.1t} = \frac{2}{3}$

$\ln(e^{-0.1t}) = \ln\left(\frac{2}{3}\right)$

$-0.1t = \ln\frac{2}{3}$

$\therefore t = \left(\frac{1}{-0.1}\right)\ln\frac{2}{3} = 4.05\ldots$

$= \boxed{4.1}$

e) $\frac{dT}{dt} = 60(-0.1)e^{-0.1t} = \boxed{-6e^{-0.1t}}$

f) $\frac{dT}{dt} = -1.8$: $\frac{-1.8}{-6} = e^{-0.1t} = 0.3$

$\Rightarrow \ln(0.3) = -0.1t$

$\Rightarrow t = \frac{-\ln(0.3)}{0.1} = 12.04$

f cont.) at $t = 12.04$, $T = 20 + 60e^{-0.1(12.04)}$

$= \boxed{38}$

Q7 i) $\dfrac{dy}{dx} = \sec^2 x - 2\sin x$

$\left(\dfrac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \dfrac{1}{\cos^2\frac{\pi}{4}} - 2\sin\frac{\pi}{4} = \boxed{2-\sqrt{2}}$

ii) $x = \tan\left(\frac{1}{2}y\right)$

$\dfrac{dx}{dy} = \frac{1}{2}\sec^2\left(\frac{1}{2}y\right) \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{1}{\frac{1}{2}\sec^2\left(\frac{1}{2}y\right)}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2}{1+\tan^2\left(\frac{1}{2}y\right)} = \boxed{\dfrac{2}{1+x^2}}$

iii) $y = e^{-x}\sin 2x$

Product rule : $\dfrac{dy}{dx} = -e^{-x}\sin 2x + 2e^{-x}\cos 2x$

$= e^{-x}(2\cos 2x - \sin 2x)$

$2\cos 2x - \sin 2x \equiv R\cos(2x+\alpha) \equiv R\cos 2x\cos\alpha - R\sin 2x\sin\alpha$

Compare coefficients : $\left.\begin{array}{l} R\cos\alpha = 2 \\ R\sin\alpha = 1 \end{array}\right\} \tan\alpha = \dfrac{\cancel{R}\sin\alpha}{\cancel{R}\cos\alpha} = \dfrac{1}{2}$ //

$\therefore \alpha = \tan^{-1}\frac{1}{2} = \boxed{0.464}$

Finding R : $R = \sqrt{2^2 + 1^2} = \sqrt{5} = \boxed{2.24}$

so $y'(x) = \boxed{e^{-x}(\sqrt{5})\cos(2x + 0.464)}$